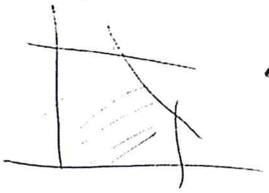
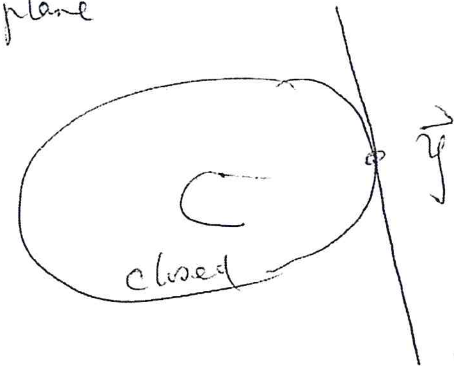
From last time



$x \geq 0,\ y \geq 0$

Thm 1.5 C closed & convex & C is bdded below.
Then every supporting hyperplane of C
has an extreme pt of C

Supporting hyperplane



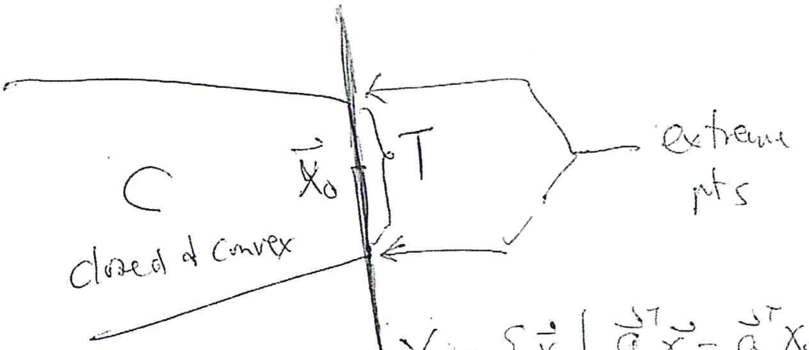
$X = \{\vec x \mid \vec c^T \vec x = \vec c^T \vec y\} \ni \vec y$

$C \subseteq X^+ \equiv \{\vec x \mid \vec c^T \vec x \underset{\phi}{\geq} \vec c^T \vec y\}$

Pf: (i) $T \neq \phi$ $\vec x_0 \in T = C \cap X \neq \phi$

(ii) ex pt of $T \Rightarrow$ ex pt of C

(iii) $\exists$ an ex pt of T



$X = \{\vec x \mid \vec a^T \vec x = \underset{\overset{\|}{z}}{\vec a^T x_0}\} \ni \bar x_0$ bdd pt

$\mathbb{R}^n \supseteq T = \{ (t_1, \dots, t_n) \in T \}$ $t_i \in \mathbb{R}$.

$$t^1 = \min_{(t_1,\dots,t_n)\in T} t_1$$

(1, 3, ...)
(1, 2, ...)
(3, 4, ...)
(1000, 3, ...)

$t^1 \rightarrow$

$T \subseteq C$
ϕ
bdded below.

T is closed & bdded from below

Case 1: t^1 is unique.
(is only one such t^1)

Case 2: $\exists$ more than 1 $\vec{t} \in T$ with t^1

$$t^2 = \min_{\substack{(t_1,..,t_n)\in T \\ t_1 = t^1}} t^2$$

Case 1 t^2 is unique (vector that ~~the~~ attains min is unique)

Case 2 t^2 is not unique

$$\boxed{t^j = \min_{\substack{(t_1,..,t_j,..t_n)\in T \\ t_1=t^1, t_2=t^2 \\ \dots t_{j-1}=t^{j-1}}} t_j}$$

process has to stop. $\mathbb{R}^n$
at jth step $j \leq n$

i.e. t^j is unique (i.e. $\exists$ only 1 vector $\vec{t^j}$ $= (\underbrace{t^1, t^2, \dots, t^j}_{\text{unique}}, \underbrace{t_{j+1} \dots t_n}_{\text{unique}}) \in T$

We claim that such $\vec{t^j}$ is an extreme pt of T   N.T.P.

$\vec{u}$ is not an extreme pt. of T

$\Rightarrow \exists\ \vec{u}_1, \vec{u}_2 \in T$

$$\vec{u} = \lambda \vec{u}_1 + (1-\lambda)\vec{u}_2 \qquad \lambda \in (0,1)$$

Proof by Contradiction.

Assume $\vec{x}^j$ is not an extreme pt ($\Rightarrow$ Contradiction)

pf: Assume $\vec{x}^j$ is not an extreme pt of T

$\exists\ \vec{y}_1, \vec{y}_2 \in T$

$$\vec{x}^j = \lambda \vec{y}_1 + (1-\lambda)\vec{y}_2 \qquad \lambda \in (0,1)$$

Let's look at the jth coordinate of the vectors

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$t^j \overset{\text{min}}{\underset{\text{unique}}{\lneqq}} (\vec{y}_1)_j \qquad \underset{t^j \lneqq}{} (\vec{y}_2)_j$$

$$t^j = \lambda a + (1-\lambda) b \qquad \lambda \in (0,1)$$

$a, b > t^j$ can't be true

$\forall \lambda, \forall a, b > t^j$

$\boxed{\text{Contradiction!}}$

$\therefore\ \vec{t}^j$ has to be an extreme pt of T

(ii) $\Rightarrow \vec{t}^j$ is also extreme pt of S

P4

Chapter 2

THEORY OF SIMPLEX METHOD

functional : functi- s.t. $f(\vec{x}) \in \mathbb{R}$

2.1 Mathematical Programming Problems

A *mathematical programming problem* is an optimization problem of finding the values of the unknown variables $x_1, x_2, \cdots, x_n$ that

$$\begin{array}{lll} \text{maximize (or minimize)} & f(x_1, x_2, \cdots, x_n) & \\ \text{subject to} & g_i(x_1, x_2, \cdots, x_n)(\leq, =, \geq) b_i, & i = 1, 2, \cdots, m \end{array} \tag{2.1}$$

where the b_i are real constants and the functions f and g_i are real-valued. The function $f(x_1, x_2, \cdots, x_n)$ is called the *objective function* of the problem (2.1) while the functions $g_i(x_1, x_2, \cdots, x_n)$ are called the *constraints* of (2.1). In vector notations, (2.1) can be written as

$$\begin{array}{lll} \text{max or min} & f(\mathbf{x}^T) & \\ \text{subject to} & g_i(\mathbf{x}^T)(\leq, =, \geq) b_i, & i = 1, 2, \cdots, m \end{array}$$

where $\mathbf{x}^T = (x_1, x_2, \cdots, x_n)$ is the solution vector.

Example 2.1. Consider the following problem.

non linear in x & y

Constrained optimization

$$\begin{array}{ll} \text{max} & f(x, y) = xy \\ \text{subject to} & x^2 + y^2 = 1 \end{array}$$

non linear in x & y

A classical method for solving this problem is the Lagrange multiplier method. Let

Unconstrained Optimization

$$L(x, y, \lambda) = xy - \lambda(x^2 + y^2 - 1). \quad \in \mathbb{R}^3 \to \mathbb{R}$$

Then differentiate L with respect to x, y, λ and set the partial derivative to 0 we get

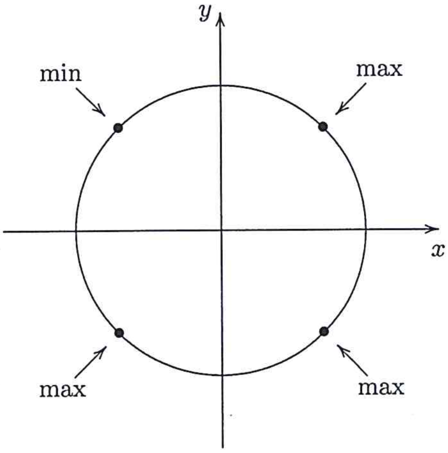
$$\frac{\partial L}{\partial x} = y - 2\lambda x = 0,$$
$$\frac{\partial L}{\partial y} = x - 2\lambda y = 0,$$
$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0.$$

The third equation is redundant here. The first two equations give

$$\frac{y}{2x} = \lambda = \frac{x}{2y}$$

P5

which gives $x^2 = y^2$ or $x = \pm y$. We find that the extrema of xy are obtained at $x = \pm y$. Since $x^2 + y^2 = 1$ we then have $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{\sqrt{2}}$. It is then easy the verify that the maximum occurs at $x = y = \frac{1}{\sqrt{2}}$ and $x = y == \frac{1}{\sqrt{2}}$ giving $f(x, y) = \frac{1}{2}$.



A *linear programming problem* (LPP) is a mathematical programming problem having a linear objective function and linear constraints. Thus the general form of an LP problem is

$x_1 \ldots x_n$

unknown variables

state

$$\begin{array}{ll} \text{maxormin} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & \left\{ \begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n (\leq,\ =,\ \geq) b_1, \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n (\leq,\ =,\ \geq) b_m, \end{array} \right. \end{array} \tag{2.2}$$

← linear

linear

Here the constants a_{ij}, b_i and c_j are assumed to be real. The constants c_j are called the *cost* or *price coefficients* of the unknowns x_j and the vector $(c_1, \cdots, c_n)^T$ is called the *cost* or *price vector*.

If in problem (2.2), all the constraints are inequality with sign $\leq$ and the unknowns x_i are restricted to nonnegative values, then the form is called *canonical.* Thus the *canonical form* of an LP problem can be written as

C.F. $\begin{array}{l} \text{min} \\ \text{max} \end{array} \ \vec{c}^{\,T}\vec{x}$

F.C.F $\left\{ \begin{array}{l} A\vec{x} \leq \vec{b} \\ \vec{x} \geq \vec{0} \end{array} \right.$ $\iff$

$\vec{b} \geq \vec{0}$

$$\begin{array}{ll} \text{maxormin} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & \left\{ \begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1, \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m, \end{array} \right. \\ \text{where} & x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{array} \tag{2.3}$$

$\vec{c}^{\,T} = (c_1, \ldots, c_n)$

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & & \cdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$

If all $b_i \geq 0$, then the form is called a *feasible canonical form.*

Before the simplex method can be applied to an LPP, we must first convert it into what is known as the *standard form:*

$$\begin{array}{ll} \text{max} & z = c_1x_1 + \cdots + c_nx_n \\ \text{subject to} & \left\{ \begin{array}{ll} a_{i1}x_1 + \cdots + a_{in}x_n = b_i, & i = 1, 2, \cdots, m \\ \qquad\qquad x_j \geq 0, & j = 1, 2, \cdots, n\,. \end{array} \right. \end{array} \tag{2.4}$$

Here the b_i are assumed to be nonnegative. We note that the number of variables may or may not be the same as before.

S.F, max $\vec{c}^{\,T}\vec{x}$

$\left\{ \begin{array}{l} A\vec{x} = \vec{b} \\ \vec{x} \geq \vec{0} \end{array} \right.$ → Feasible Region is bdd below

One can always change an LPP problem into the canonical form or into the standard form by the following procedures.

(i) If the LP as originally formulated calls for the minimization of the functional

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

we can instead substitute the equivalent objective function

$$\text{maximize} \quad z' = (-c_1)x_1 + (-c_2)x_2 + \cdots (-c_n)x_n = -z\ .$$

(ii) If any variable x_j is *free* i.e., not restricted to non-negative values, then it can be replaced by

$$x_j = x_j^+ - x_j^-, \qquad x_j^+ \geq 0 \quad x_j^- \geq 0$$

where $x_j^+ = \max(0, x_j)$ and $x_j^- = \max(0, -x_j)$ are now non-negative. We substitute $x_j^+ - x_j^-$ for x_j in the constraints and objective function in (2.2). The problem then has $(n+1)$ non-negative variables $x_1, \cdots, x_j^+, x_j^-, \cdots, x_n$.

(iii) If $b_i \leq 0$, we can multiply the i-th constraint by -1.

(iv) An equality of the form $\sum_{j=1}^n a_{ij}x_j = b_i$ can be replaced by SF

$$\sum_{j=1}^n a_{ij}x_j \leq b_i \quad \text{and} \quad \sum_{j=1}^n (-a_{ij})x_j \leq (-b_i). \qquad \text{C.F}$$

(v) Finally, any inequality constraint in the original formulation can be converted to equations by the addition of non-negative variables called the *slack* and the *surplus* variables. For example, the constraint

$$a_{i1}x_1 + \cdots + a_{ip}x_p \leq b_i \qquad \text{C.F.}$$

can be written as

$$a_{i1}x_1 + \cdots + a_{ip}x_p + x_{p+1} = b_i \qquad x_{p+1} \geq 0$$

where $x_{p+1} \geq 0$ is a slack variable. Similarly, the constraint

$$a_{j1}x_1 + \cdots + a_{jp}x_p \geq b_j$$

can be written as

$$a_{j1}x_1 + \cdots a_{jp}x_p - x_{p+2} = b_j \qquad x_{p+2} \geq 0$$

where $x_{p+2} \geq 0$ is a surplus variable. The new variables would be assigned zero cost coefficients in the objective function, i.e. $c_{p+i} = 0$.

In matrix notations, the standard form of an LPP is

a

$$\begin{array}{lll} \text{Max} & z = \mathbf{c}^T\mathbf{x} & \\ \text{subject to} & A\mathbf{x} = \mathbf{b} & (2.5) \\ \text{and} & \mathbf{x} \geq \mathbf{0} & (2.6) \end{array}$$

$\bar{x}$ satisfies, $\bar{x} \in FR$

F.S.

where A is $m \times n$, $\mathbf{b}$ is $m \times 1$, $\mathbf{x}$ is $n \times 1$ and rank $(A) = m$.

Definition 2.1. A *feasible solution* (FS) to an LPP is a vector $\mathbf{x}$ which satisfies constraints (2.5) and (2.6). The set of all feasible solutions is called the *feasible region*. A feasible solution to an LPP is said to be an *optimal solution* if it maximizes the objective function of the LPP. A feasible solution to an LPP is said to be a *basic feasible solution* (BFS) if it is a basic solution with respect to the linear system (2.5). If a basic feasible solution is non-degenerate, then we call it a *non-degenerate basic feasible solution.*

We note that the optimal solution may not be unique, but the optimum value of the problem should be unique. For LPP in feasible canonical form, the zero vector is always a feasible solution. Hence the feasible region is always non-empty.

$$\text{Min } 3x - 2y$$

$$\text{s.t.} \begin{cases} x + y \geq 3 \\ 40x + 60y = 100 \\ x \text{ is free} \\ y \geq 0 \end{cases}$$

C.F.

$\vec{C}, A, \vec{b}$

$\boxed{\text{S.F.}}$

$\vec{c}^T, A, \vec{b}$

$$\text{Min } 3x - 2y$$

$$-x - y \leq -3$$

$$40x + 60y \leq 100$$

$$\boxed{40x + 60y \geq 100}$$

$$-40x - 60y \leq -100$$

$$x = u - v$$

$$u, v, y \geq 0$$

$$\Rightarrow$$

$$\text{Min } 3u - 3v - 2y$$

$$\begin{cases} -u + v - y \leq -3 \\ 40u - 40v + 60y \leq 100 \\ -40u + 40v - 60y \leq -100 \\ u, v, y \geq 0 \end{cases}$$

$$\vec{C}^T = (3; -3, -2)$$

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 40 & -40 & 60 \\ -40 & 40 & -60 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} -3 \\ 100 \\ -100 \end{pmatrix}$$

C.F.

But not feasible

$$\max \quad -3x + 2y$$

$$\begin{cases} x + y - w = 3 & w \geq 0 \\ 40x + 60y = 100 & \\ x = u - v & u, v \geq 0 \end{cases}$$

$$y \geq 0$$

Surplus

(Same for slack)

Corresponding price coeff = 0.

$$\max \quad -3u + 3v + 2y = (-3, 3, 0, 2)\begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix}$$

$$u - v + y - w = 3$$

$$40u - 40v + 60y + 0w = 100$$

$$u, v, w, y \geq 0$$

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 40 & -40 & 60 & 0 \end{pmatrix}$$

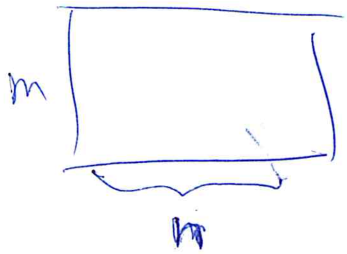
$$\vec{x} = \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 3 \\ 100 \end{pmatrix}$$

$$\vec{c}^T = (-3, 3, \underset{\substack{\uparrow \\ 0}}{\bigstar}, 2) \quad \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix}$$

Feasible solution

m<n (fat matrix)

$$\begin{cases} A\vec{x} = \vec{b} \\ \vec{x} \geq 0 \end{cases}$$



Basic solution

$$\underbrace{\left(\overset{\text{invertible}}{B_{m\times m}} \,\Big|\, R \right)}_{A} \overbrace{\begin{pmatrix} \vec{x}_B \\ \vec{x}_\beta \end{pmatrix}}^{\vec{x}} = \vec{b} \Leftrightarrow A\vec{x} = \vec{b}$$

$m-n$ $\vec{x}_\beta = 0$

$$B\vec{X}_B + \cancelto{0}{R\vec{X}_\beta} = \vec{b}$$

$$\Rightarrow \quad \vec{X}_B = B^{-1}\vec{b}$$

$$\Rightarrow \quad \vec{x} = \begin{pmatrix} B^{-1}\vec{b} \\ \vec{0} \end{pmatrix} \leftarrow \text{Basic solution}$$

BFS. ⟺ B.S. that is feasible

$$\vec{x} = \begin{pmatrix} B^{-1}\vec{b} \\ \vec{0} \end{pmatrix} \geq \vec{0}$$

Thm 2.1 FR of LPP is closed convex & bdded from below

pf: max $\vec{c}^T\vec{x}$

$$A\vec{x} = \vec{b}$$

$$\vec{x} \geq 0 \Rightarrow \text{bdd below}$$

$$\begin{pmatrix} \vec{r}_1^T \\ \vec{r}_2^T \\ \vdots \\ \vec{r}_m^T \end{pmatrix} \left(\vec{x}\right) = \vec{b}$$

$$X_1 = \{\vec{x} / \underbrace{\vec{r}_1^T\vec{x} = b_1}_{\text{hyperplane closed \& convex}}\}$$

$$\Leftrightarrow \quad \text{and} \left\{ \begin{array}{l} \vec{r}_1^T\vec{x} = b_1 \\ \vec{r}_2^T\vec{x} = b_2 \\ \vdots \\ \vec{r}_m^T\vec{x} = b_m \end{array} \right\} \text{Hyperplane}$$

Closed & Convex

$$\Downarrow$$

FR is closed & convex

Eg 2.1

$$2\ \underset{\vec{a}_1\quad\vec{a}_2\quad\vec{a}_3}{\overset{A}{\begin{pmatrix}2&1&4\\3&1&5\end{pmatrix}}}\overset{\vec{x}}{\begin{pmatrix}x_1&2\\x_2&3\\x_3&1\end{pmatrix}}=\overset{\vec{b}}{\begin{pmatrix}11\\14\end{pmatrix}}\qquad ①$$

$$\vec{x}=\begin{pmatrix}2\\3\\1\end{pmatrix}\geq\vec{0}\ ②\quad \text{is a feasible soln}$$

∵ it satisfies ①

Case 1: set $x_1=0$

$$\left(\overset{R}{\boxed{\begin{matrix}2\\3\end{matrix}}}\ \overset{B}{\boxed{\begin{matrix}1&4\\1&5\end{matrix}}}\right)\begin{pmatrix}0\\x_2\\x_3\end{pmatrix}\begin{matrix}\leftarrow x_B\\ \\ \}\vec{x}_B\end{matrix}=\begin{pmatrix}11\\14\end{pmatrix}$$

$$\Longleftrightarrow\quad\begin{pmatrix}1&4\\1&5\end{pmatrix}\begin{pmatrix}x_2\\x_3\end{pmatrix}=\begin{pmatrix}11\\14\end{pmatrix}$$

$$\Rightarrow\quad\begin{pmatrix}x_2\\x_3\end{pmatrix}=\begin{pmatrix}-1\\3\end{pmatrix}$$

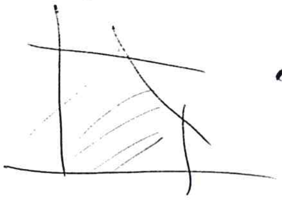
$$\Rightarrow\quad\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\-1\\3\end{pmatrix}<0\ \longrightarrow\ \text{Basic Solution}$$

not feasible

Case 2: set $x_3=0$

$$\vec{x}=\begin{pmatrix}3\\5\\0\end{pmatrix}\geq 0\quad\underline{\text{BFS}}.\quad\left(\overset{B}{\boxed{\begin{matrix}2&1\\3&1\end{matrix}}}\ \overset{R}{\boxed{\begin{matrix}4\\5\end{matrix}}}\right)$$

From last time

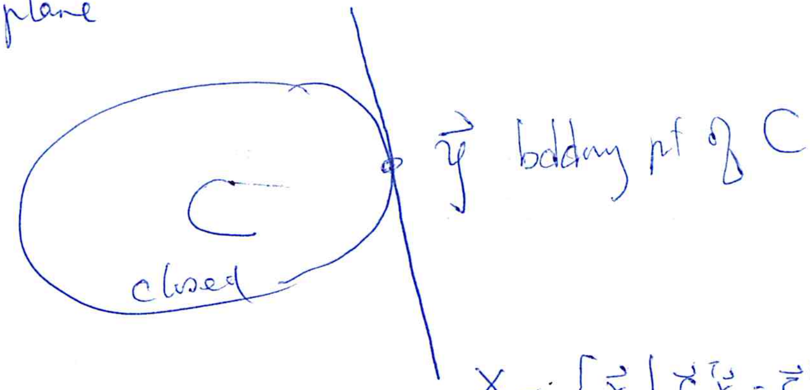


$x \ge 0, y \ge 0$

Thm 1.5 C closed & convex & C is bdded below.

Then every supporting hyperplane of C has an extreme pt of C

Supporting hyperplane



$X = \{\vec{x} \mid \vec{c}^T\vec{x} = \vec{c}^T\vec{y}\} \ni \vec{y}$

$C \subseteq X^+ \equiv \{\vec{x} \mid \vec{c}^T\vec{x} \underset{\phi}{\ge} \vec{c}^T\vec{y}\}$

Pf: (i) $T \ne \phi$ $\vec{x}_0 \in T = C \cap X \ne \phi$

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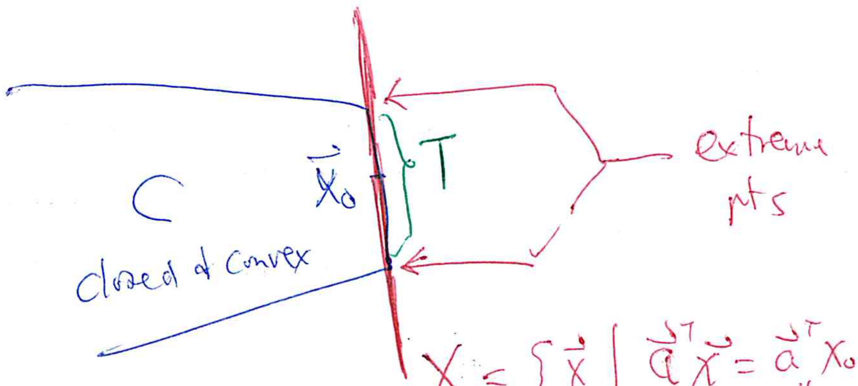

$X = \{\vec{x} \mid \vec{a}^T\vec{x} = \underset{\underset{z}{\|}}{\vec{a}^T x_0}\} \ni \vec{x}_0$ bdd pt

$$2\vec{a}_1 + 3\vec{a}_2 + 1\vec{a}_3 = \vec{b} \qquad A\vec{x}=\vec{b} \quad ①$$

$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ linear dependent

$$\boxed{1\vec{a}_1 + 2\vec{a}_2 - 1\vec{a}_3 = \vec{0}} \qquad A\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad ②$$

$$\begin{pmatrix}2&1&4\\3&1&5\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Case 1:

$$\vec{a}_1 \overset{②}{=} \vec{a}_3 - 2\vec{a}_2 \quad ③$$

③ → ① $\quad 2(\vec{a}_3 - 2\vec{a}_2) + 3\vec{a}_2 + \vec{a}_3 = \vec{b}$

$$-\vec{a}_2 + 3\vec{a}_3 = \vec{b}$$

$$(\underbrace{\vec{a}_1, \vec{a}_2, \vec{a}_3}_{A})\begin{pmatrix}0\\-1\\3\end{pmatrix} = \vec{b}$$

B.S. but not feasible

Case 2 $\quad x_3 = 0$

② → $\quad \vec{a}_3 = \vec{a}_1 + 2\vec{a}_2 \quad ④$

④ → ① $\quad 2\vec{a}_1 + 3\vec{a}_2 + \vec{a}_1 + 2\vec{a}_2 = \vec{b}$

$$3\vec{a}_1 + 5\vec{a}_2 = \vec{b}$$

$$\left(\quad A \quad\right)\begin{pmatrix}3\\5\\0\end{pmatrix} = \vec{b}$$

F+BS $\qquad \boxed{\text{BFS}}$